

A SEMIANALYTICAL BOUNDARY INTEGRAL APPROACH FOR AXISYMMETRIC ELASTIC BODIES WITH ARBITRARY BOUNDARY CONDITIONS

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Abstract—A numerical solution method for axisymmetric elastic bodies is presented. The advance is outlined in general for the case of arbitrary boundary conditions and demonstrated in detail for bending problems. To show the efficiency of the method two notch problems are calculated.

1. INTRODUCTION

In solving engineering problems numerical methods are becoming more and more important. For a few years the Boundary Element Method (BEM) has often been used besides the well-known Finite Element Method (FEM). In the meantime this integral equation approach has been developed into a powerful instrument in solving engineering problems especially potential and elasticity problems, (see Refs. [1-4] *et al.*).

In the case of linear elasticity the integral equation which is the basis of the whole procedurs, results from Betti's reciprocal work theorem:

$$\int_{\Omega} (T_{ij}u_j - U_{ij}t_j) d\Omega = 0. \quad (1.1)$$

According to this theorem, on the boundary of a region B the displacement vector u_j and the stress vector t_j of the real problem are connected with the "auxiliary" tensors U_{ij} and T_{ij} which belong to a reference group known in advance. In the BEM either the Kelvin-solution of an infinite body or the ring source solutions in the axisymmetric case, which are derived from it, are used as a reference group. The index i of the auxiliary groups U_{ij} or T_{ij} respectively, indicates the direction of force in the case of the Kelvin-solution or the kind of the ring source in the axisymmetric case. This will be explained in detail later in this paper.

The so-called source point P , indicating the location at which the force acts, is assumed to be situated on the boundary and has to be specially treated in the integration of eqn (1.1) because of the singular nature of the reference group at point P . With a suitable limit process, for instance described in [4], we obtain an integral equation of the third kind, defined by Hilbert [5]

$$C_{ij}(P)u_j(P) + \int_{\Omega} T_{ij}(P, Q)u_j(Q) d\Omega(Q) = \int_{\Omega} U_{ij}(P, Q)t_j(Q) d\Omega(Q), \quad (1.2)$$

where $C_{ij}(P)$ is a field of constants depending on the smoothness of the boundary in P and (1.2) is the starting equation of the BEM. The field point Q denotes the actual boundary point.

The integration in eqn (1.2) needs only to be done on the boundary of the domain and therefore the problem turns out to be one dimension lower than in the FEM, for example. This is an essential advantage of the described procedure because of the lower effort of the discretization and data preparation as well as in lower computer run time in most cases. After the first determination step, which yields all of the information on the boundary, a second has to be taken either by using the Somigliana formula (see [4]) or by applying the substructure technique [6] if the stress and deformation fields are required at any point of the interior.

2. THE AXISYMMETRIC DOMAIN WITH NON-AXISYMMETRIC BOUNDARY VALUES

The extension of the BEM to axisymmetric elasticity problems was investigated in [7] and, independent of it, in [8]. Also in [7] the fundamental formulation for the extension of the method to a semianalytical treatment of axisymmetric bodies with non-axisymmetric boundary values was already given without an explicit presentation of the necessary kernel functions. Later, this idea was again taken up, but also without leading to a practical procedure [9]. Our present paper consists of an explicit presentation of the method in continuation of [7]. The fundamental procedure is particularly demonstrated on the example of the bending moment transmission of a notched body of revolution.

The possibility of a semianalytical treatment of that group of problems depends on a sufficiently exact representation of the boundary values in a series of orthogonal functions of the circumferential coordinate φ . For example, let us consider a Fourier series representation of the boundary values u_j and t_j , respectively:

$$a_j = \sum_n [a_{jn}^* \cos(n\varphi) + a_{jn}^{**} \sin(n\varphi)]. \quad (2.1)$$

It can be shown with a suitable choice of the reference group and by applying the orthogonality relations (see [7]), that the integration in (1.1) in respect to the circumferential direction φ can be evaluated exactly for each term of the Fourier series. As a result, the original three-dimensional problem turns out to be a series of quasi-two-dimensional ones, which are equivalent to one-dimensional problems within the BEM because of the fact that the integration has only to be carried out along the meridian of the axisymmetric body. If only a few terms of the series (2.1) are needed, the semianalytical treatment has important advantages in contrast to real three-dimensional methods.

In the following we take up only the term $n = 1$ of the Fourier series (2.1), because of the most practical significance, which enables us to solve the bending moment as well as cross force transmission problems in a semianalytical manner. The normal force transmission is exactly the axisymmetric case, reading $n = 0$ (see [7]). Up to now the index $n = 1$ defining the first term of the Fourier series has been omitted. Moreover we use the fact that in the case of an axisymmetric body a symmetric displacement field with respect to the plane $\varphi = 0$ of a r, φ, z -cylindrical coordinate system may cause a symmetric loading system only, and also vice versa. An analogous statement holds in the antisymmetric case too. Together with the superposition principle it means, that the symmetric group as well as the antisymmetric one can be solved independently and afterwards be superposed. Therefore in the following we will deal with the symmetric case only.

The symmetric boundary value vector, following from the splitting of eqn (2.1), reads, in general

$$\mathbf{a}(r, \varphi, z) = \mathbf{e}_r a_r(r, z) \cos \varphi + \mathbf{e}_\varphi a_\varphi(r, z) \sin \varphi + \mathbf{e}_z a_z(r, z) \cos \varphi \quad (2.2)$$

where \mathbf{a} stands for \mathbf{u} or \mathbf{t} , respectively and the amplitude functions $a_r(r, z)$, $a_\varphi(r, z)$ and $a_z(r, z)$ are equal to the Fourier coefficients in eqn (2.1). The antisymmetric boundary value vector is built analogous to (2.2), only the functions $\sin \varphi$ and $\cos \varphi$ are exchanged.

As we already mentioned, in the case of axisymmetric bodies, ring source solutions were used as a reference group, if the BEM is taken as the solution method. A ring source solution has to be understood as the elastic infinite body solution, due to a line loaded circle with a radius ρ . We distinguish radial, circumferential or axial ring sources, depending on the direction of the line load. Furthermore we denote a ring source being of the class n , if the line loads vary with $\cos(n\varphi)$ or $\sin(n\varphi)$.

In the following we have only to investigate the ring source of class one. The line load vector will split up, in relation to the circumferential coordinate, into a symmetric part and a antisymmetric one.

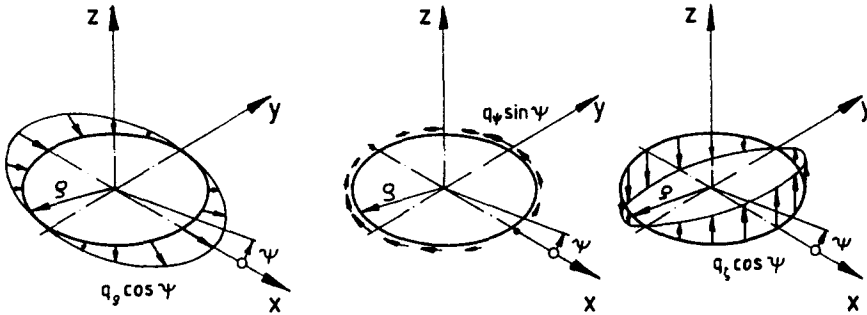


Fig. 1. Radial, circumferential and axial ring sources.

For the following derivation of the ring source solutions (see Fig. 1) it is temporarily necessary to introduce a reference coordinate system ρ, ψ, ζ . Then the symmetric line load vector reads

$$\mathbf{q}(\rho, \psi, \zeta) = \mathbf{e}_\rho q_\rho(\rho, \zeta) \cos \psi + \mathbf{e}_\psi q_\psi(\rho, \zeta) \sin \psi + \mathbf{e}_\zeta q_\zeta(\rho, \zeta) \cos \psi. \quad (2.3)$$

The ring source solution can be derived from the Kelvin solution as follows. First we determine the displacement and stress state at point Q caused by the differential force dF acting at P (see Fig. 2). Afterwards we integrate the real line load distribution of the ring source.

The local vector \mathbf{R} , connecting P and Q , reads in the r, φ, z -coordinate system

$$\mathbf{R} = \mathbf{e}_r [r - \rho \cos(\varphi - \psi)] + \mathbf{e}_\varphi \sin(\varphi - \psi) + \mathbf{e}_z \tilde{z} \quad (2.4)$$

where $\tilde{z} = z - \zeta$ and the amplitude $|\mathbf{R}| = \sqrt{(r^2 + \rho^2 - 2r\rho \cos(\varphi - \psi) + \tilde{z}^2)}$. As can be read in [10], a differential force $dF = \mathbf{e}_m dF_m$ at P causes a displacement field at Q as follows

$$du_j = \frac{1}{16\pi G(1-\nu)} \cdot \frac{1}{R} \left[(3-4\nu) dF_j + \frac{1}{R^2} R_j R_m dF_m \right], \quad j, m = r, \varphi, z \quad (2.5)$$

and also a stress field

$$d\tau_{jk} = \frac{1-2\nu}{8\pi(1-\nu)} \frac{1}{R} \left[R_m dF_m \delta_{jk} - dF_j R_k - dF_k R_j - \frac{1}{R^2} R_m dF_m R_j R_k \right], \quad j, k, m = r, \varphi, z \quad (2.6)$$

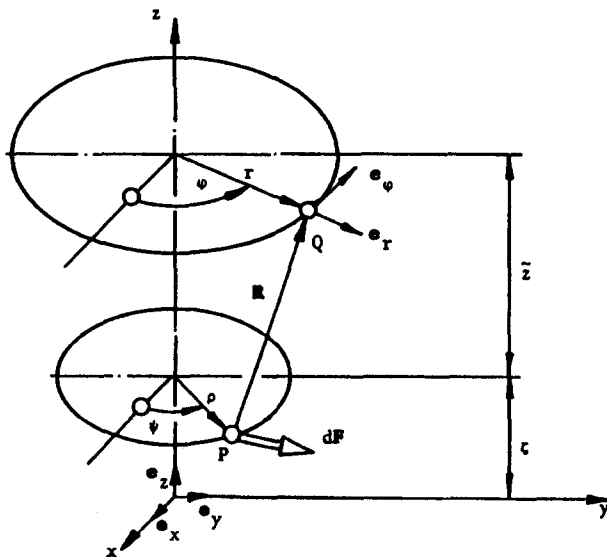


Fig. 2. Definition figure for the determination of the ring sources.

In the case of the symmetric ring source of class one, a differential line load vector acts at P as

$$d\mathbf{F} = q\rho d\psi = \mathbf{e}_\rho dF_\rho + \mathbf{e}_\psi dF_\psi + \mathbf{e}_\zeta dF_\zeta. \quad (2.7)$$

Each of the vector components $\mathbf{e}_{(i)} dF_i$ causes a special ring source state. To introduce these vector components into eqn (2.5) and (2.6), formulated in the global r, φ, z -cylinder coordinate system, each of the terms $\mathbf{e}_{(i)} dF_i$ ($i = \rho, \psi, \zeta$) has to be expressed in the global system too, namely

$$\mathbf{e}_{(i)} dF_i = \mathbf{e}_m dF_{mi} = \mathbf{e}_r dF_{ri} + \mathbf{e}_\varphi dF_{\varphi i} + \mathbf{e}_z dF_{zi}. \quad (2.8)$$

In detail this reads

$$\begin{aligned} \mathbf{e}_\rho dF_\rho &= \mathbf{e}_r q_\rho \cos\psi \cos(\varphi - \psi)\rho d\psi - \mathbf{e}_\varphi q_\rho \cos\psi \sin(\varphi - \psi)\rho d\psi \\ \mathbf{e}_\psi dF_\psi &= \mathbf{e}_r q_\psi \sin\psi \sin(\varphi - \psi)\rho d\psi + \mathbf{e}_\varphi q_\psi \sin\psi \cos(\varphi - \psi)\rho d\psi \\ \mathbf{e}_\zeta dF_\zeta &= \mathbf{e}_z q_\zeta \cos\psi \rho d\psi. \end{aligned} \quad (2.9)$$

Now introducing the load components $\mathbf{e}_{(i)} dF_i$, relating to (2.9), into eqns (2.5) and (2.6), the differential displacement and stress fields are found for each different kind of ring source. With the difference angle $\alpha = \varphi - \psi$ they can be written as

$$\begin{aligned} du_j^{(\rho)} &= -\cos(\varphi - \alpha) f_{\rho j}(\alpha) d\alpha \\ du_j^{(\psi)} &= -\sin(\varphi - \alpha) f_{\psi j}(\alpha) d\alpha \\ du_j^{(\zeta)} &= -\cos(\varphi - \alpha) f_{\zeta j}(\alpha) d\alpha \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} d\tau_{jk}^{(\rho)} &= -\cos(\varphi - \alpha) g_{\rho jk}(\alpha) d\alpha \\ d\tau_{jk}^{(\psi)} &= -\sin(\varphi - \alpha) g_{\psi jk}(\alpha) d\alpha \\ d\tau_{jk}^{(\zeta)} &= -\cos(\varphi - \alpha) g_{\zeta jk}(\alpha) d\alpha. \end{aligned} \quad (2.11)$$

The explicit expressions of the functions $f_{ij}(\alpha)$ and $g_{ijk}(\alpha)$, respectively, ($i = \rho, \psi, \zeta$) are omitted.

In the present case, where we consider only ring sources of first degree, the differential displacement and stress fields are integrated in an analytical way. In the case of higher degrees it is conjectured that it would require less effort to perform the integration in a numerical way because with increasing degree the analytical expressions for the displacement and stress fields become more and more bulky. As a result of integrating (2.10) and (2.11), respectively, along the whole circle of the ring source the finite displacement and stress fields follow, with which we can derive the auxiliary groups U_{ij} and T_{ij} , necessary in eqn (1.2).†

After integration, the reference system ρ, ψ, ζ is no longer needed. For convenience the global system is used to distinguish the different kinds of ring sources. The auxiliary groups U_{ij} may be taken immediately from the corresponding displacement fields (with $i = r, \varphi, z$ now), whereas the stress group T_{ij} is given by using Cauchy's equilibrium condition on the boundary.

With helpful abridgments the auxiliary groups are separated in φ as

$$G_{ij}(r, \varphi, z) = \hat{G}_{ij}(r, z) \cos\varphi + \bar{G}_{ij}(r, z) \sin\varphi \quad (2.12)$$

where G_{ij} can be changed to U_{ij} or T_{ij} , respectively. The expressions for \hat{G}_{ij} and \bar{G}_{ij} are listed up in the appendix. It may further be shown, that

$$\begin{aligned} \hat{G}_{ij} &\equiv 0 \quad \text{with } j = \varphi \\ \bar{G}_{ij} &\equiv 0 \quad \text{with } j = r, z. \end{aligned} \quad (2.13)$$

†The authors thank Mr. Dipl.-Ing. W. Escher for his useful help in determining the auxiliary groups.

Therefore it is clear, that the functions G_{ij} are symmetric functions as well as the boundary values or the line load vector, respectively.

With the above, Betti's theorem (1.1) delivers the relation ($i = r, \varphi, z$) with respect to the three kinds of ring sources)

$$\int_{\Omega_s} \int_{\varphi=0}^{2\pi} [(T_{ir}u_r - U_{ir}t_r) \cos \varphi + (T_{i\varphi}u_\varphi - U_{i\varphi}t_\varphi) \sin \varphi + (T_{iz}u_z - U_{iz}t_z) \cos \varphi] d\varphi r(s) ds = 0 \quad (2.14)$$

where Ω_s is the meridian of the axisymmetric body and ds is a differential line element on Ω_s . Considering (2.13), in (2.14) the integration with respect to φ can be carried out separately, which, with the abridgment

$$G_{ij}^* = \begin{cases} \hat{G}_{ij} & \text{with } j \neq \varphi \\ \check{G}_{ij} & \text{with } j = \varphi \end{cases} \quad (2.15)$$

leads to the simple relation

$$\int_{\Omega_s} (T_{ij}^* u_j - U_{ij}^* t_j) d\Omega_s = 0 \quad (2.16)$$

where $d\Omega_s = 2\pi r(s) ds$. In the following step from Betti's theorem to the starting equation of the BEM, analogous limit processes as in the axisymmetric case have to be carried out, leading to

$$C_{ij}^*(P) u_j(P) + \int_{\Omega_s} T_{ij}^*(P, Q) u_j(Q) d\Omega_s = \int_{\Omega_s} U_{ij}^*(P, Q) t_j(Q) d\Omega_s \quad (2.17)$$

The field of constants $C_{ij}^*(P)$ must not be explicitly known but may be eliminated as shown in the next chapter. Information about the interior of the domain can again be obtained with either the Somigliana formula or with the substructure technique.

3. NUMERICAL TREATMENT

Equation (2.7) can be solved for a general case only in a numerical way. For a convenient representation of the boundary values a normal-tangential-coordinate system will be used in the following. The transformed terms are indicated by an apostrophe. Furthermore, the meridian, on which the integration has to be done, is represented piecewise by simple functions (straight line, circle, parabola) and divided in n intervals. Within such a boundary element the magnitudes u'_j of the boundary displacements as well as the magnitudes t'_j of the boundary tractions are expressed by polynomial functions

$$x'_j(\zeta) = \sum_{m=1}^q M^m(\zeta) x_j'^m \quad x'_j = u'_j, t'_j \quad (3.1)$$

with the intrinsic boundary coordinate ζ , where $-1 \leq \zeta \leq 1$. The summation of the products of shape functions $M^m(\zeta)$ and node values $x_j'^m$ has to be carried out to the degree $q-1$ of approximation (see[6, 11]). In the procedure presented here linear, quadratic, and cubic polynomial functions are alternatively possible.

With (3.1) eqn (2.17) reads in the discretized manner (see[11])

$$\begin{aligned} C_{ij}^*(P) u'_j(P) + \sum_{l=1}^n \sum_{m=1}^q [u_j'^m \int_{s_l} T_{ij}^*(P, \zeta) M^m(\zeta) J(\zeta) r(\zeta) d\zeta] \\ = \sum_{l=1}^n \sum_{m=1}^q [t_j'^m \int_{s_l} U_{ij}^*(P, \zeta) M^m(\zeta) J(\zeta) r(\zeta) d\zeta] \end{aligned} \quad (3.2)$$

where $J(\zeta)$ is the well-known Jacobi function and s_1 is the l th integration interval. Because of the subdivision of the meridian into n intervals, the boundary integral turns out to be a sum of n parts where each may be calculated separately according to the polynomial representation of the boundary value amplitudes. The terms u_j^{ml} or t_j^{ml} , respectively, are the boundary value amplitudes of the m th node within the l th boundary element.

For each location of the source point we get three scalar equations with (3.2). If P sequentially takes the place of all node points of the meridian, according to the given boundary conditions we get a system of linear equations for the determination of the unknown boundary value amplitudes u_j^{ml} and t_j^{ml} , which reads with the matrix formulation

$$A\mathbf{u} = B\mathbf{t}. \quad (3.3)$$

Here the vector \mathbf{u} contains all the node displacement components u_j^{ml} and the vector \mathbf{t} contains the node traction components t_j^{ml} . The elements of the square matrices A and B result from the eqn (3.2) by integrating along the actual boundary element.

If P is not situated in the actual interval of integration the Gauss quadrature formula is used. If P and Q are in identical positions the integrant becomes singular. Then the diagonal elements of the matrix B may still be calculated with a modified Gauss formula, because the singularity is of a logarithmic order only, as can be shown easily. The diagonal elements of the matrix A , however, have higher order singularities and therefore they have to be treated in a special way. As an analogy to [6], we proceed as follows. Simple solutions in elasticity with non-axisymmetric boundary value distributions are used which deliver all necessary information at the boundary of the axisymmetric body under investigation. Such solutions can be rigid body motions or may be taken from the well-known solutions of the bi-potential equation. In this way these solutions deliver completely known vectors \mathbf{u}^* and \mathbf{t}^* for determining the unknown diagonal elements of the matrix A .

Therefore an integration of higher order singularities as well as the explicit knowledge of the coefficients $C_{ij}^*(P)$ are no longer necessary.

After this the matrix equation (3.3) has to be rearranged depending on the boundary value problem and then it may be solved with a usual Gauss algorithm. In the end all displacement and traction amplitudes of each node are known. The stress amplitudes in the normal or circumferential direction, respectively, which are not taken directly from (3.3), can be determined afterwards with the equilibrium condition at the boundary, the kinematic relations and Hooke's law.

4. EXAMPLES

To demonstrate the efficiency of the described method two examples are investigated. First we take a cylinder with a spherical cavity under a pure bending moment. For a ratio $\rho/R = 0.3$ (Fig. 3) the convergence of the stress concentration factor α_k versus the computer run time is

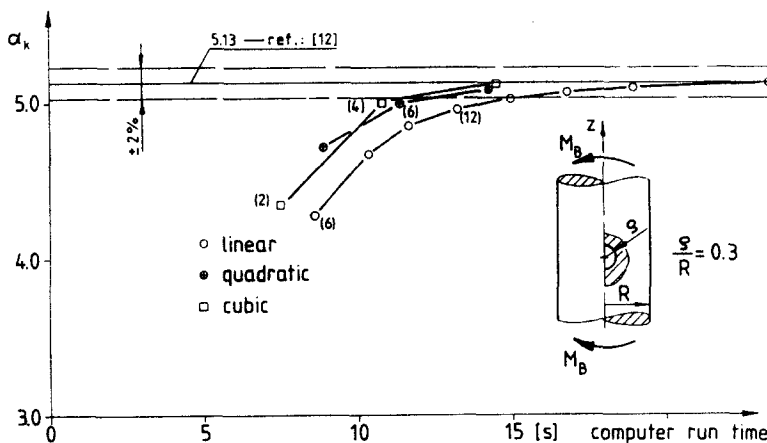


Fig. 3. Cylinder with a spherical cavity under pure bending.

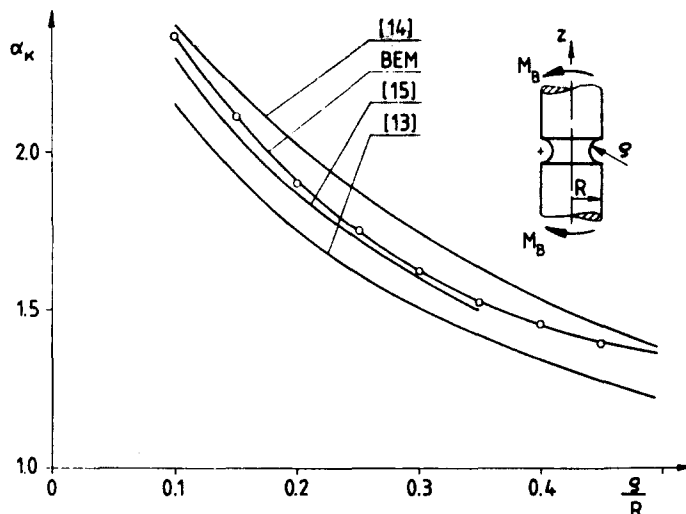


Fig. 4. Cylinder with a circumferential notch under pure bending.

shown.† Thereby a linear, quadratic or cubic degree of approximation was used sequentially. The values in brackets point to the number of intervals along the notch. It is obvious that the convergence is quite good. As to the degree of approximation of the boundary value functions no remarkable difference is evident in this simple example.

In a second example a cylinder with a circumferential notch (see Fig. 4) loaded with a pure bending moment is investigated. In Fig. 4 the stress concentration factor, taken from the BEM, is compared with results given by Neuber [13], by Rühl [14] or by the diagrams of Peterson [15]. It is obvious that the values taken from the BEM are in good agreement with those of Peterson.

In both examples the stress and displacement fields in the interior of the domain are not determined. As earlier mentioned, however, this may easily be done either pointwise with Somigliana's formula or with the substructure technique.

5. CONCLUDING REMARKS

A general procedure for axisymmetric bodies with non-axisymmetric boundary values is demonstrated. The major advantage is the reducing of an otherwise three-dimensional problem to a sequence of one-dimensional ones. The method is outlined especially for problems with boundary values, which may be represented by the first term of a Fourier-series. A computer program is written in which the meridian can be approximated by straight lines, circles or parabolae, where as for the approximation of the boundary displacements and tractions linear, quadratic or cubic polynomial functions alternatively may be chosen. The method delivers sufficient results as can be seen from the examples.

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†The computations were performed by a CYBER 175 computer at the Leibniz Rechenzentrum der Bayerischen Akademie der Wissenschaften.

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APPENDIX

Displacement fields of ring sources:

$$\begin{aligned} \dot{U}_{rr} &= \frac{1}{4Gr^2\rho^2C_8} \left\{ \left[-\frac{C_4}{3}[(11-16\nu)C_1 + (17-16\nu)\bar{z}^2] + \frac{C_6}{C_5}(C_7^2 - \bar{z}^4) \right] \mathbf{E} \right. \\ &\quad \left. + \frac{4}{3} [C_6[2(1-2\nu)C_1 + (5-4\nu)\bar{z}^2] + 2(5-4\nu)r^2\rho^2] \mathbf{K} \right\} \\ \dot{U}_{r\theta} &= -\frac{1}{3Gr^2\rho^2C_8} (C_4[(5-4\nu)C_3 + 2(1-2\nu)\rho^2] \mathbf{E} + \{2(7-8\nu)r^2\rho^2 - C_6[(5-4\nu)C_3 + 2(1-2\nu)\rho^2]\} \mathbf{K}) \\ \dot{U}_{rz} &= \frac{\bar{z}}{2Gr\rho^2C_8} \left\{ \left[C_4 + \frac{C_6}{C_5}(C_7 + \bar{z}^2) \right] \mathbf{E} - 2C_3 \mathbf{K} \right\} \\ \dot{U}_{\theta r} &= \frac{1}{2Gr^2\rho^2C_8} \left\{ -\frac{2}{3} C_4[2(1-2\nu)r^2 + (5-4\nu)C_2] \mathbf{E} + \left[\frac{1}{3}(7-8\nu)C_4C_5 - C_6(C_7 - \bar{z}^2) \right] \mathbf{K} \right\} \\ \dot{U}_{\theta\theta} &= \frac{1}{3Gr^2\rho^2C_8} \{ -(5-4\nu)C_4C_6 \mathbf{E} + [(5-4\nu)C_6^2 + 4(1-2\nu)r^2\rho^2] \mathbf{K} \} \\ \dot{U}_{\theta z} &= \frac{\bar{z}}{Gr\rho^2C_8} (C_4 \mathbf{E} - C_6 \mathbf{K}) \\ \dot{U}_{zr} &= \frac{\bar{z}}{2Gr^2\rho C_8} \left\{ - \left[C_4 - \frac{C_6}{C_5}(C_7 - \bar{z}^2) \right] \mathbf{E} + 2C_2 \mathbf{K} \right\} \\ \dot{U}_{z\theta} &= \frac{\bar{z}}{Gr^2\rho C_8} (-C_4 \mathbf{E} + C_6 \mathbf{K}) \\ \dot{U}_{zz} &= \frac{1}{Gr\rho C_8} \left\{ \left[-(3-4\nu)C_4 + \frac{C_6}{C_5}\bar{z}^2 \right] \mathbf{E} + [(3-4\nu)C_1 + 2(1-2\nu)\bar{z}^2] \mathbf{K} \right\}. \end{aligned}$$

Stress fields of ring sources:

$$\begin{aligned} \bar{T}_{rr} &= \frac{1}{4r^3\rho^2C_8} \left\{ \left\langle \left[\frac{2}{3} C_4[-2(1+4\nu)r^2 + (13-20\nu)\rho^2 + 2(11-10\nu)\bar{z}^2] + \frac{C_6}{C_5}[4(1-2\nu)(C_2^2 + r^2(\bar{z}^2 - \rho^2)) \right. \right. \right. \\ &\quad \left. \left. + 48r^2\rho^2 - 6C_6(4r^2 + \rho^2 - 2\bar{z}^2)] + 24r^4\frac{C_1}{C_3} - \frac{4C_6^2}{C_4C_5}[8r^2\rho^2C_2 + C_6(\bar{z}^4 + C_7C_1 - 2r^2(2\rho^2 + \bar{z}^2))] \right] \right\rangle n_r \\ &\quad - 2r\bar{z} \left\{ (5-4\nu)C_4 - \frac{1}{C_5}[12r^2C_3 + C_6((1+4\nu)\rho^2 - (11-4\nu)C_3)] + \frac{C_6^2}{C_4C_5^2} 4(C_7^2 - \bar{z}^4) \right\} n_z \mathbf{E} \\ &\quad + \left\langle \left[\frac{1}{3}(7-8\nu)C_4C_5 - 4(1-2\nu)(r^2 + 2\rho^2 + 4\bar{z}^2) + 3C_2^2 \right] + 6r^2(4r^2 + 5\rho^2 + 6\bar{z}^2) \right. \\ &\quad \left. - 18(C_3^2 + \rho^2(2r^2 + \bar{z}^2)) + \frac{C_6}{C_4C_5}[8r^2\rho^2C_2 + C_6(\bar{z}^4 + C_7C_1 - 2r^2(2\rho^2 + \bar{z}^2))] \right] \mathbf{K} \right\rangle n_r \\ &\quad + 2r\bar{z} \left\{ (7-8\nu)C_1 + (13-8\nu)\bar{z}^2 + \frac{C_6}{C_4C_5}(C_7^2 - \bar{z}^4) \right\} n_z \mathbf{K} \right\} \\ \bar{T}_{r\theta} &= -\frac{1}{4r^3\rho^2C_8} \left\langle 2 \left\{ \left[C_4(3\rho^2 - \bar{z}^2) - \frac{1}{3}(7-8\nu)(r^2 + 4C_2) \right] + \frac{C_6}{C_5}(C_7^2 - \bar{z}^4) \right\} n_r \right. \\ &\quad \left. + 2r\bar{z} \left\{ (5-4\nu)C_4 + \frac{C_6}{C_5}(C_7 + \bar{z}^2) \right\} n_z \right\rangle \mathbf{E} + \left\langle \left[\frac{1}{3}(5-16\nu)C_4C_5 + 2C_6[4(1-2\nu)C_2 + 9\bar{z}^2] + C_7^2 - \bar{z}^4 \right] n_r \right. \\ &\quad \left. + 8r\bar{z}[\rho^2 - (3-2\nu)C_6] n_z \right\rangle \mathbf{K} \right\rangle \end{aligned}$$

$$\begin{aligned} \hat{T}_r &= \frac{1}{2r^2\rho^2C_8} \left\langle \left\langle \bar{z} \left\{ -(5-4\nu)C_4 + \frac{C_6}{C_5}[(1+4\nu)\rho^2 - (11-4\nu)C_3] + 12r^2\frac{C_3}{C_5} - \frac{C_6^2}{C_4C_5^2}4(C_6^2 - \bar{z}^4) \right\} n_r \right. \right. \\ &\quad + 2r \left\{ (1-2\nu)C_4 + \frac{C_6}{C_5}(1-2\nu)(C_7 + \bar{z}^2) + 6\bar{z}^2\frac{C_3}{C_5} - 4\bar{z}^2\frac{C_6^2}{C_4C_5^2}(C_7 + \bar{z}^2) \right\} n_z \Big\rangle \mathbf{E} \\ &\quad + \left\langle \bar{z} \left\{ (7-8\nu)C_1 + (13-8\nu)\bar{z}^2 + \frac{C_6}{C_4C_5}(C_7^2 - \bar{z}^4) \right\} n_r \right. \\ &\quad \left. + 2e \left\{ -2(1-2\nu)r^2 - (5-4\nu)\bar{z}^2 + \frac{C_6}{C_4C_5}\bar{z}^2(C_7 + \bar{z}^2) \right\} n_z \right\rangle \mathbf{K} \Big\rangle \\ \hat{T}_{\theta r} &= -\frac{1}{2r^2\rho^2C_8} \left\langle \left\langle \left\{ \frac{1}{3}C_4[(11+8\nu)r^2 - (37-32\nu)C_2] - \frac{C_6}{C_5}(C_7 + \bar{z}^2)^2 \right\} n_r \right. \right. \\ &\quad + 2r\bar{z} \left\{ (5-4\nu)C_4 + \frac{C_6}{C_5}(C_7 + \bar{z}^2) \right\} n_z \Big\rangle \mathbf{E} + 4 \left\langle \left\{ -r^2[2(2-\nu)r^2 + \frac{2}{3}(11-7\nu)\rho^2 + (5-2\nu)\bar{z}^2] + \frac{2}{3}(5-4\nu)C_6^2 \right\} n_r \right. \\ &\quad \left. - r\bar{z}\{(3-2\nu)C_2 + 2(1-\nu)r^2\}n_z \right\rangle \mathbf{K} \Big\rangle \\ \hat{T}_{\theta\theta} &= \frac{1}{r^2\rho^2C_8} \left\langle \left\langle \left\{ \frac{1}{3}C_4(5-7\nu)r^2 + (17-13\nu)C_2 \right\} - \frac{C_6^2}{C_5}(C_7 - \bar{z}^2)(1-\nu) \right\} n_r \right. \\ &\quad - 2r\bar{z} \left\{ (2-\nu)C_4 + \frac{C_6}{C_5}(1-\nu) \right\} n_z \Big\rangle \mathbf{E} + \left\langle \left\{ -\frac{1}{3}(2-\nu)C_4C_5 + C_6[\nu r^2 - (6-5\nu)C_2] \right\} n_r \right. \\ &\quad \left. + 2r\bar{z}(3-2\nu)C_6n_z \right\rangle \mathbf{K} \Big\rangle \\ \hat{T}_{\theta z} &= \frac{1}{r^2\rho^2C_8} \left\langle \left\langle -\bar{z} \left\{ (5-4\nu)C_4 + \frac{C_6}{C_5}(C_7 + \bar{z}^2) \right\} n_r + 2r \left\{ (1-2\nu)C_4 + \frac{C_6}{C_5}\bar{z}^2 \right\} n_z \right. \right. \\ &\quad \left. \left. + 2\{\bar{z}[2(1-\nu)r^2 + (3-2\nu)C_2]n_r - r[(1-2\nu)C_1 + 2(1-\nu)\bar{z}^2]n_z\} \right\rangle \mathbf{K} \right\rangle \\ \hat{T}_{rz} &= \frac{1}{2r^2\rho C_8} \left\langle \left\langle \bar{z} \left\{ 3C_4 + 5\frac{C_6^2}{C_5} - \frac{4}{C_5}r^2[2(1+\nu)r^2 + (5+2\nu)C_2] + \frac{C_6^2}{C_4C_5^2}16r^2\bar{z}^2 \right\} n_r \right. \right. \\ &\quad + 2r \left\{ (1-2\nu)C_4 + \frac{C_6}{C_5}(1-2\nu)(C_7 + \bar{z}^2) - \frac{C_2}{C_5}6\bar{z}^2 + \frac{C_6^2}{C_4C_5^2}4\bar{z}^2(C_7 + \bar{z}^2) \right\} n_z \Big\rangle \mathbf{E} \\ &\quad + \left\langle 4\bar{z} \left\{ 2(r^2\nu - C_2) - \frac{C_6}{C_4C_5}r^2\bar{z}^2 \right\} n_r + 2r \left\{ -2(1-2\nu)\rho^2 + (1+4\nu)\bar{z}^2 - \frac{C_6}{C_4C_5}\bar{z}^2(C_7 + \bar{z}^2) \right\} n_z \right\rangle \mathbf{K} \Big\rangle \\ \hat{T}_{z\theta} &= \frac{1}{r^2\rho C_8} \left\langle \left\langle \bar{z} \left\{ 3C_4 + \frac{C_6}{C_5}(C_7 + \bar{z}^2) \right\} n_r + 2r \left\{ (1-2\nu)C_4 - \frac{C_6}{C_5}\bar{z}^2 \right\} n_z \right. \right. \\ &\quad \left. \left. - 2\{\bar{z}(r^2 + 2C_2)n_r + r(C_1 - 2\nu C_6)n_z\} \right\rangle \mathbf{K} \right\rangle \\ \hat{T}_{zz} &= \frac{1}{r^2\rho C_8} \left\langle \left\langle \left\{ (1-2\nu)C_4 - \frac{C_6}{C_5}(1-2\nu)(C_7 - \bar{z}^2) - \frac{C_2}{C_5}6\bar{z}^2 + \frac{C_6^2}{C_4C_5^2}4\bar{z}^2(C_7 + \bar{z}^2) \right\} n_r \right. \right. \\ &\quad - 2r\bar{z} \left\{ \frac{1}{C_5}[(1-2\nu)C_6 - 3\bar{z}^2] + \frac{C_6^2}{C_4C_5^2}4\bar{z}^2 \right\} n_z \Big\rangle \mathbf{E} - \left\langle \left\{ 2(1-2\nu)\rho^2 - (1+4\nu)\bar{z}^2 + \frac{C_6}{C_4C_5}\bar{z}^2(C_7 + \bar{z}^2) \right\} n_r \right. \\ &\quad \left. - 2r\bar{z} \left\{ (1-2\nu) + \frac{C_6}{C_4C_5}\bar{z}^2 \right\} n_z \right\rangle \mathbf{K} \Big\rangle \end{aligned}$$

with ν = Poisson's ratio; G = shear modulus; $\mathbf{K}(\pi/2, \kappa)$, $\mathbf{E}(\pi/2, \kappa)$ complete elliptic integrals; β = boundary normal angle; $n_r = \cos\beta$; $n_z = \sin\beta$; $C_1 = r^2 + \rho^2$; $C_2 = \rho^2 + \bar{z}^2$; $C_3 = r^2 + \bar{z}^2$; $C_4 = (r + \rho)^2 + \bar{z}^2$; $C_5 = (r - \rho)^2 + \bar{z}^2$; $C_6 = r^2 + \rho^2 + \bar{z}^2$; $C_7 = r^2 - \rho^2$; $C_8 = 16r^2(1 - \nu)\sqrt{(r + \rho)^2 + \bar{z}^2}$; $\kappa = \sqrt{(4r\rho)/[(r + \rho)^2 + \bar{z}^2]}$.